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# Optimal Placement and Gains of Sensors and Actuators for Feedback Control

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**This paper presents an optimal design method for placement and gains of actuators and sensors in output feedback control systems. A quadratic performance function is minimized using nonlinear programming. The new contribution is the derivation of analytical expressions for the gradients of the performance function. These gradients are important because they can be efficiently computed and do not suffer from errors associated with finite differences. This makes the method suitable for systems with a large number of actuators. Several numerical studies are performed for a two-dimensional structure. In all cases, even with 100 optimization variables, convergence is rapid.**

## Nomenclature

| | |
|---|---|
| $a, b$ | = dimensions of membrane structure |
| $\boldsymbol{b}(\boldsymbol{x}_a)$ | = $n \times m$ actuator placement matrix |
| $\boldsymbol{C}_d$ | = $n \times n$ modal damping matrix |
| $c_0$ | = damping per unit area |
| $\boldsymbol{c}(\boldsymbol{x}_s)$ | = $r \times n$ sensor placement matrix |
| $\mathbf{diag}[\cdot]$ | = diagonal matrix with diagonal elements given in brackets |
| $\boldsymbol{F}$ | = $m \times r$ time-invariant output feedback gain matrix |
| $\boldsymbol{I}$ | = $n \times n$ identity matrix |
| $m$ | = number of actuators |
| $m_0$ | = mass per unit area |
| $n$ | = number of modes in structural model |
| $\boldsymbol{Q}$ | = weight matrix for structural response |
| $\boldsymbol{R}$ | = weight matrix for control forces |
| $r$ | = number of sensors |
| $t$ | = time |
| $tr[\cdot]$ | = trace of matrix |
| $\boldsymbol{u}(t)$ | = $m$ vector of control forces |
| $w(\boldsymbol{x}, t)$ | = displacements |
| $\boldsymbol{x}$ | = spatial coordinate |
| $\boldsymbol{x}_a$ | = actuator placement coordinate |
| $\boldsymbol{x}_s$ | = sensor placement coordinate |
| $\boldsymbol{y}(t)$ | = $r$ vector of sensor outputs |
| $\alpha_j, \beta_j$ | = parameters for open-loop modes |
| $\delta_{ij}$ | = Kronecker delta function |
| $\boldsymbol{\Phi}(\boldsymbol{x})$ | = $n$ vector of eigenfunctions |
| $\Lambda$ | = $n \times n$ modal stiffness matrix |
| $\lambda$ | = elastic constant |
| $\omega_i$ | = $i$th natural frequency of open-loop system |
| $*$ | = optimal values |

## Introduction

THERE are two sets of design parameters in feedback control systems for flexible structures: the gains and placement of sensors and actuators. Although there are well-established methods for determining optimal gains,[1] methods for optimal placement are relatively new. An early optimal placement method minimized control energy; the design of the feedback gains was considered separately.[2–5] Schulz and Heimbold[6] developed a method for concurrent design of both placement and gains. The method is optimal in that it maximizes energy dissipation due to control action. The optimization problem is solved by a gradient-based nonlinear programming technique. More recently, the energy-based approach has been solved by Chen et al. using simulated annealing,[7] and Rao et al. using genetic algorithms.[8] An entirely different approach to the optimal placement problem, which maximizes certain eigenproperties of the system, has been proposed by Johnson et al.[9]

In the first part of the present paper, the method of Schulz and Heimbold[6] is extended by using a performance function that includes both the structural response and the control effort. Then, analytical expressions are derived for the gradients of the performance function. The gradient expressions are of fundamental importance in nonlinear programming because they avoid the two counteracting errors associated with finite differences: truncation error, which is proportional to the finite difference interval, and condition error, which is inversely proportional to the interval.[10,11] In addition, the computational effort in evaluating the gradient expressions is far smaller than that required in finite differences, which makes the optimization method efficient. The derivation of the gradient expressions is based on the solution of a Lyapunov equation and the use of Bellman's expansion[12] and Kleinman's lemma.[13]

To avoid dependence on initial conditions for the disturbances, the trace of the performance function, originally proposed by Levine and Athans,[13] is used. The resulting performance function is an average over all initial conditions that can be represented by a unit vector in the state space. Thus, although the method is useful for structures that may have a wide variety of external disturbances, it may be inappropriate for structures with specific disturbances, such as those generated by attitude controllers.[14] Nevertheless, the efficiency of the method makes it suitable for systems with a large number of actuators. This is illustrated by an example numerical study of a two-dimensional structure with 1, 11, and 100 actuators.

## Equations of Motion

### Physical Model of System

The vibration response of an elastic structure subjected to an applied force $F(\boldsymbol{x}, t)$ is given by

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$$\left(m_0\frac{\mathrm{d}^2}{\mathrm{d}t^2}+c_0\frac{\mathrm{d}}{\mathrm{d}t}+\lambda L\right)w(\boldsymbol{x},t)=F(\boldsymbol{x},t) \tag{1}$$

The first two terms are for the inertial and viscous damping forces. The third term is the elastic restoring force, expressed in terms of a linear differential operator $L$. If Eq. 1 is solved by separation of variables, the result is of the form

$$w(\boldsymbol{x},t)=\boldsymbol{\phi}^T(\boldsymbol{x})\boldsymbol{q}(t) \tag{2}$$

Here, the $\boldsymbol{q}(t)$ are generalized coordinates given by the solution of

$$\boldsymbol{I}\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}_d\dot{\boldsymbol{q}}(t)+\boldsymbol{\Lambda}\boldsymbol{q}(t)=\boldsymbol{b}\boldsymbol{u}(t) \tag{3}$$

where the applied force is expressed as a product of the $n\times m$ matrix $\boldsymbol{b}$ and an $m$ vector $\boldsymbol{u}(t)$.

Consider $m$ point-force actuators located at $\boldsymbol{x}_{aj}$, where $j=1,\dots,m$, and $r$ velocity sensors located at $\boldsymbol{x}_{sp}$, where $p=1,\dots,r$. Then, $\boldsymbol{u}(t)$ is the vector of actuator forces, and

$$\boldsymbol{b}(\boldsymbol{x}_a)=[\boldsymbol{\phi}(\boldsymbol{x}_{a1})\cdots\boldsymbol{\phi}(\boldsymbol{x}_{am})] \tag{4}$$

For direct output velocity feedback control, the $r$-dimensional measurement vector $\boldsymbol{y}(t)$ is given by

$$\boldsymbol{y}(t)=\boldsymbol{c}(\boldsymbol{x}_s)\dot{\boldsymbol{q}}(t) \tag{5}$$

where $\boldsymbol{c}(\boldsymbol{x}_s)$ is the $m\times r$ placement matrix

$$\boldsymbol{c}(\boldsymbol{x}_s)=\begin{bmatrix}\boldsymbol{\phi}^T(\boldsymbol{x}_{s1})\\ \vdots \\ \boldsymbol{\phi}^T(\boldsymbol{x}_{sr})\end{bmatrix} \tag{6}$$

The control force is proportional to the output measurement

$$\boldsymbol{u}(t)=-\boldsymbol{F}\boldsymbol{y}(t) \tag{7}$$

where $F$ is an $m\times r$ time-invariant gain matrix.

### State Equations

It is convenient to express Eqs. (3–7) as a first-order state equation:

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}_0\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t),\qquad \boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t) \tag{8}$$

where

$$\boldsymbol{A}_0=\begin{bmatrix}0 & \boldsymbol{I}\\ -\boldsymbol{\Lambda} & -\boldsymbol{C}_d\end{bmatrix},\qquad \boldsymbol{x}(t)=\begin{bmatrix}\boldsymbol{q}(t)\\ \dot{\boldsymbol{q}}(t)\end{bmatrix}$$
$$\boldsymbol{B}=\begin{bmatrix}0\\ \boldsymbol{b}\end{bmatrix},\qquad \boldsymbol{C}=[0\quad \boldsymbol{c}] \tag{9}$$

The free-vibration response of the closed-loop system for any initial condition $\boldsymbol{x}(0)$ is[1,15]

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\boldsymbol{x}(0) \tag{10}$$

where $e^{\boldsymbol{A}t}$ is the fundamental transition matrix and

$$\boldsymbol{A}=\boldsymbol{A}_0-\boldsymbol{BFC} \tag{11}$$

## Optimization Problem

In the following, an optimal design procedure is developed for the actuator placement $\boldsymbol{x}_{aj}$, sensor placement $\boldsymbol{x}_{sp}$, and feedback gains $\boldsymbol{F}$. First, a performance function is chosen that includes both the structural response and the control effort. The standard performance function is

$$\hat{J}=\frac{1}{2}\int_0^\infty(\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})\,\mathrm{d}t \tag{12}$$

which can be expanded in terms of the fundamental transition matrix:

$$\hat{J}=\boldsymbol{x}^T(0)\left[\frac{1}{2}\int_0^\infty e^{\boldsymbol{A}^Tt}\boldsymbol{Q}+\boldsymbol{C}^T\boldsymbol{F}^T\boldsymbol{RFC})e^{\boldsymbol{A}t}\,\mathrm{d}t\right]\boldsymbol{x}(0) \tag{13}$$

A design procedure that uses this performance function will require specific values for the initial state $\boldsymbol{x}(0)$.[6] Herein, this dependence on $\boldsymbol{x}(0)$ is eliminated by using an average performance function proposed by Levine and Athans.[13] The initial state is modeled as a random vector uniformly distributed on the surface of the $2n$-dimensional unit sphere. It has been shown that the average, or expected, value of $\hat{J}$, scaled by $2n$, is

$$J=\frac{1}{2}\int_0^\infty \mathrm{tr}[e^{\boldsymbol{A}^Tt}(\boldsymbol{Q}+\boldsymbol{C}^T\boldsymbol{F}^T\boldsymbol{RFC})e^{\boldsymbol{A}t}]\,\mathrm{d}t \tag{14}$$

This performance function, which is explicitly in terms of the actuator and sensor placement vectors $\boldsymbol{x}_{aj}$ and $\boldsymbol{x}_{sp}$ and the feedback gain matrix $\boldsymbol{F}$, is used hereafter. The optimization problem is

$$\min_{\boldsymbol{x}_a,\boldsymbol{x}_s,\boldsymbol{F}} J(\boldsymbol{x}_a,\boldsymbol{x}_s,\boldsymbol{F})\rightarrow \boldsymbol{x}_a^*,\boldsymbol{x}_s^*,\boldsymbol{F}* \tag{15}$$

subject to constraints

$$\boldsymbol{x}_a\subset \boldsymbol{X}_a,\qquad \boldsymbol{x}_s\subset\boldsymbol{X}_s \tag{16}$$

where $\boldsymbol{X}_a$ and $\boldsymbol{X}_s$ are subsets of the domain of the structure. For simplicity, the subscripts $j$ and $p$ are dropped. The optimization problem can be solved with standard nonlinear programming techniques. Direct-search methods are the simplest but are restricted to a small number of optimization variables. Descent methods can be applied to problems with a large number of variables[10] but require the gradients of the performance function. As stated in the introduction, the descent method will encounter numerical difficulties if finite differences are used to compute the gradients. To avoid these difficulties, analytical expressions for the gradients, presented in the next section, will be used.

## Algorithm

### Transformation to Lyapunov Equations

To determine the gradients of the performance function, the following two matrix integrals are needed:

$$\boldsymbol{K}=\int_0^\infty e^{\boldsymbol{A}^Tt}\left(\boldsymbol{Q}+\boldsymbol{C}^T\boldsymbol{F}^T\boldsymbol{RFC}\right)e^{\boldsymbol{A}t}\,\mathrm{d}t \tag{17}$$

$$\boldsymbol{L}=\int_0^\infty e^{\boldsymbol{A}t}e^{\boldsymbol{A}^Tt}\,\mathrm{d}t \tag{18}$$

The matrix $\boldsymbol{K}$ is related to the performance function by

$$J=\tfrac{1}{2}\mathrm{tr}[\boldsymbol{K}] \tag{19}$$

The matrices $\boldsymbol{K}$ and $\boldsymbol{L}$ can be obtained without numerical integration by solving the associated Lyapunov equations[13]

$$\boldsymbol{KA}+\boldsymbol{A}^T\boldsymbol{K}+\boldsymbol{Q}+\boldsymbol{C}^T\boldsymbol{F}^T\boldsymbol{RFC}=0 \tag{20}$$

$$\boldsymbol{LA}^T+\boldsymbol{AL}+\boldsymbol{I}=0 \tag{21}$$

The Lyapunov equations are efficiently solved by the method of Bartels and Stewart.[16]

### Gradients of Performance Function

For the optimal design problem, the gradients with respect to the control gains $\boldsymbol{F}$, actuator placement matrix $\boldsymbol{B}$, and sensor placement matrix $\boldsymbol{C}$ are needed. These gradients are evaluated using Bellman's expansion and Kleinman's Lemma. The derivation for the first gradient was derived by Levine and Athans[13]; the derivations for the latter two gradients are given in the Appendix. The final results are given in the following.

The gradient with respect to the control gains $\boldsymbol{F}$ is

$$\frac{\partial J}{\partial \boldsymbol{F}}=\boldsymbol{RFCLC}^T-\boldsymbol{B}^T\boldsymbol{KLC}^T \tag{22}$$

This gradient simplifies for certain special cases. If the system is collocated, i.e., the sensors and actuators are at the same locations, then $m=r$ and $\boldsymbol{C}=\boldsymbol{B}^T$. Such systems are stable if $\boldsymbol{F}$ is symmetric

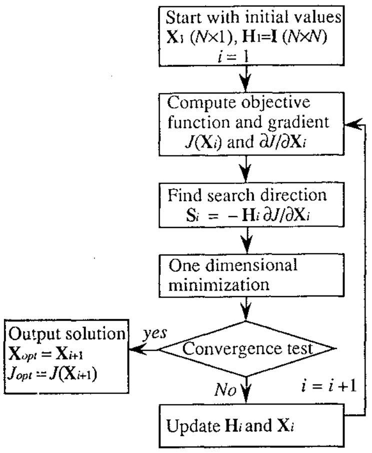


**Fig. 1 Flowchart of DFP algorithm.**

and positive definite.[6] Using the Cholesky decomposition, $\boldsymbol{F}$ can be written as

$$\boldsymbol{F} = \boldsymbol{G}\boldsymbol{G}^T \tag{23}$$

where $\boldsymbol{G}$ is a lower triangular matrix with positive diagonal elements. The gradient of the performance function with respect to $\boldsymbol{G}$ is given by the chain rule:

$$\frac{\partial J}{\partial \boldsymbol{G}} = \left[\frac{\partial J}{\partial \boldsymbol{F}} + \left(\frac{\partial J}{\partial \boldsymbol{F}}\right)^T\right]\boldsymbol{G} \tag{24}$$

If the system is locally controlled, i.e., it is collocated and the gain matrix is of the form $\boldsymbol{F} = \mathbf{diag}\{fi^2\}$, then the gradient simplifies to

$$\frac{\partial J}{\partial f_i} = 2f_i\frac{\partial J}{\partial F_{ii}} \tag{25}$$

where $\partial J/\partial F_{ii}$ are the diagonal elements of Eq. (22).

The gradients with respect to the actuator and sensor placement matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are

$$\frac{\partial J}{\partial \boldsymbol{B}} = -\boldsymbol{K}\boldsymbol{L}\boldsymbol{C}^T\boldsymbol{F}^T \tag{26}$$

$$\frac{\partial J}{\partial \boldsymbol{C}} = -\boldsymbol{F}^T\boldsymbol{B}^T\boldsymbol{K}\boldsymbol{L} + \boldsymbol{F}^T\boldsymbol{R}\boldsymbol{F}\boldsymbol{C}\boldsymbol{L} \tag{27}$$

If the system is collocated, then, since $\boldsymbol{C} = \boldsymbol{B}^T$, it is necessary to sum the preceding two gradients to yield:

$$\frac{\partial J}{\partial \boldsymbol{B}} = -\boldsymbol{L}\boldsymbol{K}\boldsymbol{B}\boldsymbol{F} + \boldsymbol{L}\boldsymbol{B}\boldsymbol{F}^T\boldsymbol{R}\boldsymbol{F} - \boldsymbol{K}\boldsymbol{L}\boldsymbol{B}\boldsymbol{F}^T \tag{28}$$

The gradients with respect to the coordinate vectors $\boldsymbol{x}_a$ are $\boldsymbol{x}_s$ are subsequently obtained by the chain rule.

#### Numerical Optimization

The optimal design problem in Eqs. (15) and (16) is a constrained nonlinear programming problem. However, as long as the subspaces $\boldsymbol{X}_a$ and $\boldsymbol{X}_s$ are polygons or other well-defined geometries, the constrained problem can be mathematically transformed to an unconstrained problem. This point is illustrated in the example analysis. Therefore, the optimization problem presented herein can be treated as an unconstrained minimization problem.

One of the most robust gradient-based unconstrained optimization techniques is the Davidon-Fletcher-Powell (DFP) algorithm,[10] which is summarized by the flowchart in Fig. 1. The basic parameters for each iteration $i$ are the vector of optimization variables $\boldsymbol{X}_i$, an approximate inverse of the Hessian matrix $\boldsymbol{H}_i$, and a search direction vector $\boldsymbol{S}_i$. After the search direction vector is computed, the one-dimensional minimization proceeds as follows. An accelerated step-size algorithm is used to determine an interval in the search direction that contains at least one local minimum. Then, within this interval, cubic interpolation is used to find a minimum. For a well-defined convex performance function, convergence is rapid. However, if there are more than one local minima in the searching direction, convergence may be slow; therefore, a combination of the cubic-interpolation and direct-root methods is used. If convergence does not occur after several iterations, then the direct-root method is used to reduce the domain so that it includes only one local minimum. In the last step of the algorithm, the following convergence tests are used:

$$\left|\frac{J(\boldsymbol{X}_{i+1}) - J(\boldsymbol{X}_i)}{J(\boldsymbol{X}_i)}\right| \le \varepsilon_1 \tag{29}$$

$$\left|\frac{\partial J}{\partial X_j}\right| \le \varepsilon_2, \qquad j = 1, \ldots, N \tag{30}$$

Here, $\varepsilon_1$ and $\varepsilon_2$ are convergence control parameters and $X_j$ is the $j$th component of vector $\boldsymbol{X}_i$.

## Examples

The example structure is a rectangular membrane, with dimensions $a = 1.00$ and $b = 1.03$, as shown in Fig. 2. Although plates, shells, or other continuous structures could have been used, the membrane was chosen because its lower natural frequencies are relatively closely spaced. This structural-dynamic feature makes the optimal design problem particularly challenging.

The derivation of the equations of motion can be found in standard texts.[17] The open-loop modal damping ratios are 0.005 for all modes; a total of 11 modes are considered in the numerical analysis. The mode shapes of the open-loop system are

$$\phi_j(x, y) = \sin\left(\alpha_j\frac{x}{a}\right)\sin\left(\beta_j\frac{y}{b}\right) \tag{31}$$

The modal parameters $\alpha_j$ and $\beta_j$ and the natural frequencies are given in Table 1.

The output feedback control system consists of $m$ point-force actuators and collocated sensors located at $\{x_j, y_j\}$, where $j = 1, \ldots, m$. This example control system was chosen because it is readily visualized. The matrix $b$ defined in Eq. (4) is

$$\boldsymbol{b} = \begin{bmatrix} \phi_1(x_1, y_1) & \cdots & \phi_1(x_m, y_m) \\ \vdots & & \vdots \\ \phi_n(x_1, y_1) & \cdots & \phi_n(x_m, y_m) \end{bmatrix} \tag{32}$$

For the performance function in Eq. (14), the following weighting matrices are used:

$$\boldsymbol{Q} = \begin{bmatrix} \Lambda & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}, \qquad \boldsymbol{R} = R\begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \tag{33}$$

The first term in the performance function is

$$J_E = \frac{1}{4n}\int_0^\infty \mathrm{tr}[\boldsymbol{\Phi}^T(t, 0)\boldsymbol{Q}\boldsymbol{\Phi}(t, 0)]\,\mathrm{d}t \tag{34}$$

which is the time integral of the structure's vibration energy. The second term in the performance function is the time integral of the control energy.

The constraints for actuator placement are the boundaries of the membrane, i.e.,

$$0 \le x_i \le a, \qquad 0 \le y_i \le b, \qquad i = 1, \ldots, m \tag{35}$$

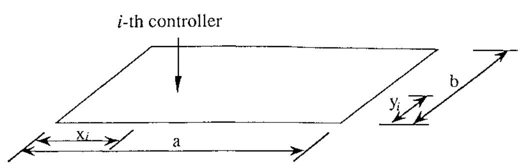


**Fig. 2 Membrane with feedback control system.**

**Table 1 Modal properties of open-loop system**

| Case number | Frequencies | $\alpha_j$ | $\beta_j$ |
|---|---|---|---|
| 1 | 1.394 | $\pi$ | $\pi$ |
| 2 | 2.184 | $\pi$ | $2\pi$ |
| 3 | 2.223 | $2\pi$ | $\pi$ |
| 4 | 2.788 | $2\pi$ | $2\pi$ |
| 5 | 3.080 | $\pi$ | $3\pi$ |
| 6 | 3.153 | $3\pi$ | $\pi$ |
| 7 | 3.533 | $2\pi$ | $3\pi$ |
| 8 | 3.574 | $3\pi$ | $2\pi$ |
| 9 | 4.010 | $\pi$ | $4\pi$ |
| 10 | 4.116 | $4\pi$ | $\pi$ |
| 11 | 4.181 | $3\pi$ | $3\pi$ |

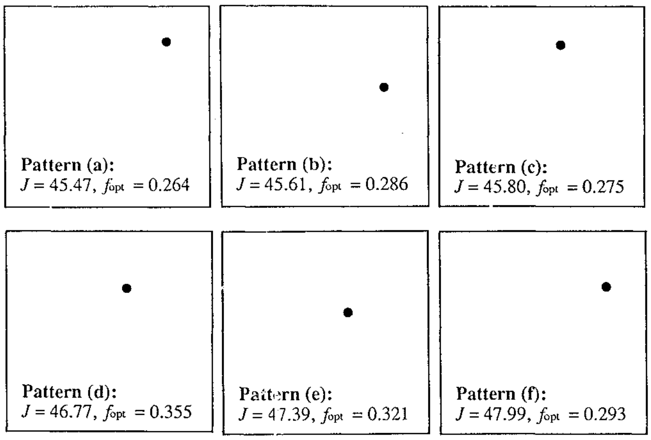


**Fig. 3 Placements for single-controller case.**

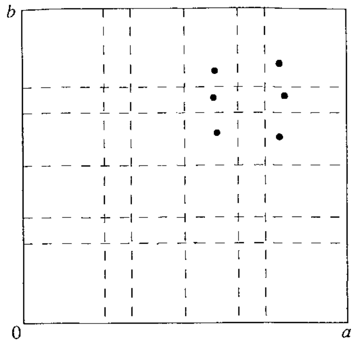


**Fig. 4 Superimposed placements for the single-controller case: (—) nodal lines for open-loop modes.**

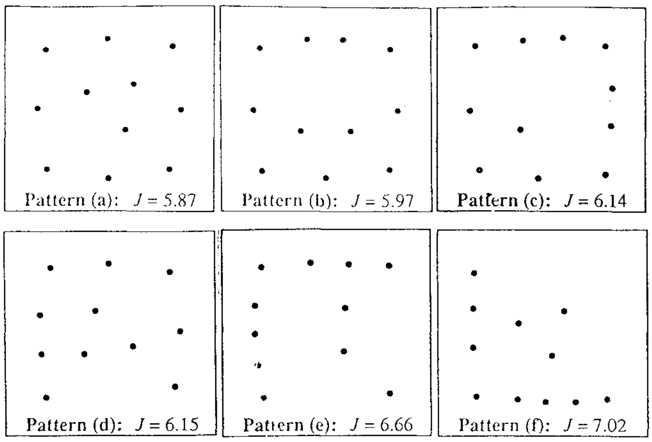


**Fig. 5 Placements for 11-controller case.**

To eliminate these mathematical constraints, the constrained coordinates $x_j$ and $y_j$ are transformed to unconstrained coordinates $\eta_j$ and $\xi_j$ by

$$x_i = a \sin^2(\eta_i), \qquad y_i = b \sin^2(\xi_i), \qquad i = 1, \ldots, m \tag{36}$$

Although there are an infinite number of values for $\eta_j$ and $\xi_j$ for each pair of values for $x_j$ and $y_j$ the algorithm is well behaved for any initial values for the optimization variables. The gradients of the performance function with respect to the new coordinates, $\eta_j$ and $\xi_j$, are readily obtained from the chain rule.

In the following, three numerical studies are presented: optimal placement and gains for 1 and 11 actuators and optimal gains for 100 actuators. Single-loop feedback control gains are used, making the gain matrix diagonal.

### Optimal Placement and Gains for Single Actuator/Sensor

The simplest optimal design problem is for a single actuator and collocated sensor. There are only three optimization variables: the velocity gain and the $x$ and $y$ coordinates of the controller. The control penalty is $R = 10$, which yields control with relatively high authority. If the coordinates of the controller are fixed, there is only a single value for the velocity gain that minimizes the performance function. However, with the three optimization variables, there are many local minima. To obtain as many different local minima as possible, the nonlinear programming algorithm is executed with 20 computer-generated random initial values for the optimization variables.

Accounting for symmetry, six distinct solutions were obtained. The placement of the actuators and the corresponding values for the performance function $J$ and gains $f_{\rm opt}$ are shown in Fig. 3. Solution a, which yields the lowest performance value and lowest gain, is the global optimum; 10 of the 20 initial values for the optimization variables converged to this solution. Solution f yields the largest performance value, and only one initial value converged to this local minima. Solution d yields the largest gain. The difference between the smallest and largest performance values is only 5%, but the difference between the smallest and largest gains is 34%. Thus, for this problem, the multiple local minima yield nearly equivalent performance values.

The six placement results are superimposed in Fig. 4; due to symmetry, they are plotted only in the upper right portion of the structure. The nodal lines of the open-loop mode shapes are also indicated by dashed lines. The figure shows that the placement of the actuators lies away from the nodal lines, as expected from controllability theory.[18] Between the six placement results, there are three regions bounded by nodal lines that are empty. In each of these regions, local minima exist; however, they are considerably higher than the remaining six local minima. The proposed algorithm is robust in that it never converges to these local minima.

### Optimal Placement and Gains for Eleven Actuators/Sensors

In this case, the number of actuators is equal to the number of modes in the structural model. There are 33 optimization variables, 3 for each controller. As in the previous case, the control penalty is $R = 10$. The nonlinear programming algorithm is executed with 14 computer-generated random initial values for the optimization variables.

Accounting for symmetry, six distinct solutions were obtained. The placement of the actuators and the corresponding values for the performance function $J$ are shown in Fig. 5. Solution a yields the smallest value, $J = 5.87$, which, as expected, is considerably lower than the value for a single controller. Solution f yields the largest value, $J = 5.87$, which differs from solution a by 20%. Figure 5 shows that solution a has the most evenly distributed placement pattern. Some of the other solutions have placements that are concentrated on one side of the structure. For all cases, the gains of the single-loop actuators are close to 0.2.

The 14 placement results are superimposed in Fig. 6, and, as in Fig. 4, the nodal lines of the open-loop mode shapes are indicated by dashed lines. The actuator placements are clustered in certain nearly symmetric locations about the membrane. Some of the actu-

**Table 2 Eigenvalues of system**

| Mode number | Real part | Imaginary part |
|---|---|---|
| 1 | −0.3766 | 1.3419 |
| 2 | −0.3810 | 2.1506 |
| 3 | −0.3814 | 2.1902 |
| 4 | −0.3814 | 2.7613 |
| 5 | −0.3837 | 3.0556 |
| 6 | −0.3855 | 3.1294 |
| 7 | −0.3857 | 3.5124 |
| 8 | −0.3873 | 3.5525 |
| 9 | −0.3897 | 3.9912 |
| 10 | −0.3909 | 4.0975 |
| 11 | −0.3905 | 4.1630 |

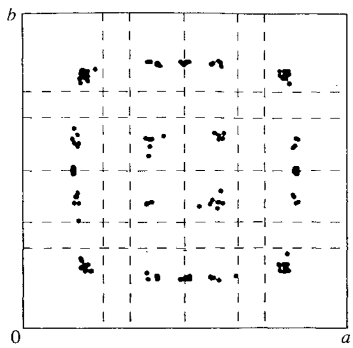


**Fig. 6 Superimposed placements for 11-controller case: (—) nodal lines for open-loop modes.**

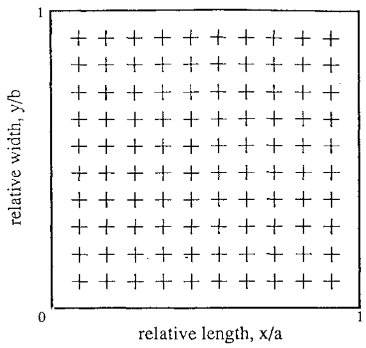


**Fig. 7 Placement of 100 controllers.**

ator locations lies on nodal lines; these actuators are still effective since several of the open-loop modes do not have nodal lines at these locations.

#### Optimal Gains for One Hundred Actuators/Sensors

The placement of controllers is shown in Fig. 7. Since the actuators are uniformly placed throughout the structure, optimal placement is not considered. The control penalty is $R = 100$, which yields control with relatively low authority. This optimization problem is a simple study of distributed control, which has wide potential applications.[19] The problem is also a demonstration of how easy the nonlinear programming algorithm can handle 100 variables.

Since the analytical expression for the gradient in Eq. (11) is used, the algorithm is efficient and the convergence is fast. For convergence control parameters $\varepsilon_1 = \varepsilon_2 = 0.0001$ in Eqs. (29) and (30), only 30 iterations were required. The final optimal result for the gains is plotted as a function over the membrane in Fig. 8. The gain levels are symmetric with respect to the structure geometry and are nearly constant. This indicates that for distributed control, spatially constant gains may be an appropriate design strategy. The eigenvalues of the closed-loop system are given in Table 2. The real parts of all of the eigenvalues are nearly constant, indicating that the optimal solution follows the uniform damping criteria.[20]

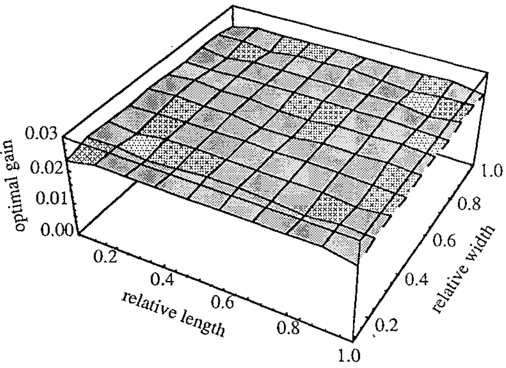


**Fig. 8 Optimal gains for 100-controller case.**

## Conclusion

A new optimal design method is presented for placement and gains of actuators and sensors in feedback control systems. The method extends the method developed by Levine and Athans[13] by solving for optimal placement. The method also enhances the nonlinear programming approach developed by Schulz and Heimbold[6] with the use of analytical expressions for the gradients of the performance function. The method is efficient and can handle a large number of optimization variables. Several numerical studies were performed for a two-dimensional structure. Multiple local minima for the performance function are possible for the optimal placement problem, and randomly generated initial values for the optimization variables were used to obtain a set of solutions. In all cases, even with 100 optimization variables, convergence was rapid.

## Appendix: Derivation of Gradients of Performance Function

To derive analytical expressions for the gradients of the performance function, Bellman's expansion and Klein's lemma are needed.

#### Bellman's Expansion[12]

Let $S$ and $T$ be square matrices. Then the exponential matrix function can be expressed as

$$e^{(S+\varepsilon T)t} = e^{St} + \varepsilon \int_0^t e^{S(t-s)} T e^{Ss}\, ds + \mathcal{O}(\varepsilon^2) \tag{A1}$$

where $\varepsilon$ is small relative to unity.

#### Kleinman's Lemma[13]

Consider a scalar function $f(S)$ of an $r \times n$ matrix $S$ with the following property:

$$f(S + \varepsilon\, \Delta S) - f(S) = \varepsilon\, \mathrm{tr}[M(S)\, \Delta S] \tag{A2}$$

where $M(S)$ is an $n \times r$ matrix. Then, the gradient of $f$ is

$$\frac{df(S)}{dS} = M^T(S) \tag{A3}$$

#### Gradient with Respect to Actuator Placement Matrix B

The following derivation is a minor modification of that given by Levine and Athans[13] for the gradient with respect to the control gain matrix $F$. For an infinitesimal change in $B$, the performance function is given by

$$J(B + \varepsilon\, \Delta B) = \frac{1}{2}\, \mathrm{tr} \int_0^\infty \left[ e^{(A-\varepsilon\Delta BFC)^T t} [Q + C^T F^T RFC]\, e^{(A-\varepsilon\Delta BFC)t} \right] dt \tag{A4}$$

According to Bellman's expansion in Eq. (A1), the exponential matrix functions can be expanded as

$$e^{(A-\varepsilon\Delta BFC)t} = e^{At} - \varepsilon I_0 + \mathcal{O}(\varepsilon^2) \tag{A5}$$

$$e^{(A-\varepsilon\Delta BFC)^T t} = e^{A^T t} - \varepsilon I_0^T + \mathcal{O}(\varepsilon^2) \tag{A6}$$

where

$$I_0 = \int_0^t e^{A(t-s)} \Delta \boldsymbol{B F C} e^{As} \, \mathrm{d}s \tag{A7}$$

The increment of the performance function, $\Delta J$, is obtained by combining Eqs. (A4–A6) and subtracting Eq. (14):

$$\Delta J = J(\boldsymbol{B} + \Delta \boldsymbol{B}) - J(\boldsymbol{B}) = \varepsilon \,\mathrm{tr}\left( \int_0^\infty [-e^{A^T t}(\boldsymbol{Q} + \boldsymbol{C}^T \boldsymbol{F}^T \boldsymbol{RFC}) \boldsymbol{I}_0] \, \mathrm{d}t \right) + \mathcal{O}(\varepsilon^2) \tag{A8}$$

Substituting $\boldsymbol{I}_0$ into Eq. (A8) yields

$$\Delta J = \varepsilon \,\mathrm{tr}\left( - \int_0^\infty \int_0^t e^{A^T t}(\boldsymbol{Q} + \boldsymbol{C}^T \boldsymbol{F}^T \boldsymbol{RFC}) \times e^{A(t-s)} \Delta \boldsymbol{BFC} e^{As} \, \mathrm{d}s \, \mathrm{d}t \right) + \mathcal{O}(\varepsilon^2) \tag{A9}$$

Changing the order of integration and using the transformation of variables $\tau = t\text{–}s$ yield

$$\Delta J = -\varepsilon \,\mathrm{tr}\left( \int_0^\infty \int_0^\infty e^{A^T(\tau+s)} + \left(\boldsymbol{Q} + \boldsymbol{C}^T \boldsymbol{F}^T \boldsymbol{RFC}\right) \times e^{A\tau} \Delta \boldsymbol{BFC} e^{As} \, \mathrm{d}s \, \mathrm{d}\tau \right) + \mathcal{O}(\varepsilon^2) \tag{A10}$$

Using the commutative property of matrix multiplication within the trace operator and substituting Eqs. (17) and (18), yield the simplified result

$$\Delta J = -\varepsilon \,\mathrm{tr}(\boldsymbol{FCLK} \, \Delta \boldsymbol{B}) + \mathcal{O}(\varepsilon^2) \tag{A11}$$

Finally, Kleinman's lemma is used to obtain the gradient with respect to $\boldsymbol{B}$. The result is given in Eq. (26).

**Gradient with Respect to Sensor Placement Matrix $\boldsymbol{C}$**

For an infinitesimal change in $\boldsymbol{C}$, the performance function is given by

$$J(\boldsymbol{C} + \varepsilon \, \Delta \boldsymbol{C}) = \frac{1}{2} \,\mathrm{tr} \int_0^\infty \left\{ e^{(A - \varepsilon BF \Delta C)^T t} [\boldsymbol{Q} + (\boldsymbol{C} + \varepsilon \, \Delta \boldsymbol{C})^T \times \boldsymbol{F}^T \boldsymbol{RF}(\boldsymbol{C} + \varepsilon \, \Delta \boldsymbol{C})] e^{(A - \varepsilon BF \Delta C)t} \right\} \mathrm{d}t \tag{A12}$$

As in the preceding derivation for the gradient with respect to $\boldsymbol{B}$, Bellman's expansion, an integral transformation, and the commutative property in the trace operator are successively applied to obtain the simplified result

$$\Delta J = \varepsilon \,\mathrm{tr}[\boldsymbol{L}(-\boldsymbol{KB} + \boldsymbol{CF}^T \boldsymbol{R}) \boldsymbol{F} \, \Delta \boldsymbol{C}] + \mathcal{O}(\varepsilon^2) \tag{A13}$$

Once again, Kleinman's lemma is used to obtain the gradient with respect to $\boldsymbol{C}$. The result is given in Eq. (27).

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